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901 Homework 2

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Problem 2.6.13

If $\{B_k\}$ are events such that

$$\sum_{k=1}^{n} P(B_k) > n-1,$$

then show $P\left(\bigcap_{k=1}^{n} B_k\right) > 0$.

Solution: Suppose by way of contrapositive that $P\left(\bigcap_{k=1}^{n} B_k\right) = 0$. Then,

$$1 = P\left(\bigcup_{k=1}^{n} B_k^c\right) \leq \sum_{k=1}^{n} P(B_k^c) = n - \sum_{k=1}^{n} P(B_k).$$

Therefore, we have $\sum_{k=1}^{n} P(B_k) \leq n-1$. This proves the result.

Problem 3.4.8

Let X and Y be random variables and let $A \in \mathcal{B}$. Prove that the function

$$Z(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in A, \\ Y(\omega), & \text{if } \omega \in A^c \end{cases}$$

is a random variable.

Solution: Since X and Y are random variables, we have $X\colon (\Omega, \mathcal{B}) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ and $Y\colon (\Omega, \mathcal{B}) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, i.e. for $C \in \mathcal{B}(\mathbb{R})$, $X^{-1}(C) \in \mathcal{B}$ and $Y^{-1}(C) \in \mathcal{B}$. We must show this is also true for Z. Take $C \in \mathcal{B}(\mathbb{R})$ and notice

$$\begin{aligned} Z^{-1}(C) &= \{\omega \in \Omega\colon Z(\omega) \in C\} = \{\omega \in \Omega\colon Z(\omega) \in C, \omega \in A\} \cup \{\omega \in \Omega\colon Z(\omega) \in C, \omega \in A^c\} \\ &= \{\omega \in A\colon X(\omega) \in C\} \cup \{\omega \in A^c\colon Y(\omega) \in C\} \\ &= \Big(A \cap \{\omega \in \Omega\colon X(\omega) \in C\}\Big) \cup \Big(A^c \cap \{\omega \in \Omega\colon Y(\omega) \in C\}\Big) \in \mathcal{B}. \end{aligned}$$

Therefore, Z is a random variable.

Problem 3.4.11

Let $(\Omega, \mathcal{B}, P)$ be $\big([0,1], \mathcal{B}([0,1]), \lambda\big)$, where λ is the Lebesgue measure on $[0,1]$. Define the process $\{X_t, 0 \le t \le 1\}$ by

$$X_t(\omega) = \begin{cases} 0, & \text{if } t \neq \omega, \\ 1, & \text{if } t = \omega. \end{cases}$$

Show that each X_t is a random variable. What is the σ-field generated by $\{X_t, 0 \le t \le 1\}$?

> **Solution:** Recall that $\mathcal{B}\big([0,1]\big)$ can be generated by any kind of intervals in $[0,1]$. Now, choose $t \in [0,1]$ and see that
>
> $$X_t^{-1}\big(\{0\}\big) = [0,t) \cup (t,1] \in \mathcal{B}\big([0,1]\big)$$
>
> and also
>
> $$X_t^{-1}\big(\{1\}\big) = \{t\} = \bigcap_{n=1}^{\infty} [t, t+1/n] \in \mathcal{B}\big([0,1]\big).$$
>
> Therefore, X is a random variable. For any specific t_0,
>
> $$\sigma(X_{t_0}) = X_{t_0}^{-1}\big(\mathcal{B}([0,1])\big) = X_{t_0}^{-1}\Big(\big\{\{0\},\{1\}\big\}\Big) = \big\{\emptyset, \Omega, \{t_0\}, [0,t_0) \cup (t_0,1]\big\}.$$
>
> Since the process is over all $0 \le t \le 1$, we conclude the σ-field generated by this process is
>
> $$\sigma\big(\{X_t, 0 \le t \le 1\}\big) = \mathcal{B}\big([0,1]\big).$$

Problem 3.4.18

Coupling. If X and Y are random variables on $(\Omega, \mathcal{B})$, show

$$\sup_{A \in \mathcal{B}} \big|P(X \in A) - P(Y \in A)\big| \le P(X \neq Y).$$

> **Solution:** Let $A \in \mathcal{B}$ be arbitrary. Notice
>
> $$\{\omega \in \Omega\colon X(\omega) \in A\} \subset \{\omega \in \Omega\colon Y(\omega) \in A\} \cup \{\omega \in \Omega\colon X(\omega) \neq Y(\omega)\}.$$
>
> Therefore, we have
>
> $$P(X \in A) \le P(Y \in A) + P(X \neq Y)$$
>
> and so $P(X \in A) - P(Y \in A) \le P(X \neq Y)$. Similarly,
>
> $$\{\omega \in \Omega\colon Y(\omega) \in A\} \subset \{\omega \in \Omega\colon X(\omega) \in A\} \cup \{\omega \in \Omega\colon X(\omega) \neq Y(\omega)\}.$$
>
> Thus, we also have
>
> $$P(Y \in A) \le P(X \in A) + P(X \neq Y)$$
>
> and so $P(Y \in A) - P(X \in A) \le P(X \neq Y)$. Combining this with the inequality shown above, we conclude
>
> $$\big|P(X \in A) - P(Y \in A)\big| \le P(X \neq Y).$$
>
> Since A was arbitrary, we can take supremum on both sides with respect to $A \in \mathcal{B}$ to obtain the result.

Problem 3.4.19

Suppose $T\colon (\Omega_1, \mathcal{B}_1) \to (\Omega_2, \mathcal{B}_2)$ is a measurable mapping and X is a random variable on Ω_1. Show $X \in \sigma(T)$ iff there is a random variable Y on $(\Omega_2, \mathcal{B}_2)$ such that for all $\omega_1 \in \Omega_1$, $X(\omega_1) = Y\big(T(\omega_1)\big)$.

Solution: We first show the backwards direction. Assume there is a random variable Y on $(\Omega_2, \mathcal{B}_2)$ such that for all $\omega_1 \in \Omega_1, X(\omega_1) = Y\big(T(\omega_1)\big)$. Then,

$$\sigma(X) = \{X^{-1}(B)\colon B \in \mathcal{B}(\mathbb{R})\} = \Big\{T^{-1}\big(Y^{-1}(B)\big)\colon B \in \mathcal{B}(\mathbb{R})\Big\} \subset \sigma(T)$$

and so $X \in \sigma(T)$. Now for the forward direction, suppose $X \in \sigma(T)$. First, assume X is a simple random variable, i.e. $X = \sum_{k=1}^n a_k I_{A_k}$, where each $A_k \in \sigma(T)$ and $a_k \in \mathbb{R}$. Again, since each $A_k \in \sigma(T)$, there exists $B_k \in \mathcal{B}_2$ such that $T^{-1}(B_k) = A_k$. Define $Y = \sum_{k=1}^n a_k I_{B_k}$. Clearly Y is a random variable since it is the sum of random variables and also for $\omega_1 \in \Omega_1$,

$$Y\big(T(\omega_1)\big) = \sum_{k=1}^n a_k I_{B_k}\big(T(\omega_1)\big) = \sum_{k=1}^n a_k I_{T^{-1}(B_k)}(\omega_1) = \sum_{k=1}^n a_k I_{A_k}(\omega_1) = X(\omega_1).$$

Continuing the buildup, consider X to be a nonnegative random variable. Recall we showed in class that for a nonnegative random variable, whether extended or not, there exists a sequence of simple random variables $X_1, X_2, \ldots$ such that $X_n \uparrow X$. Define $X_i = \sum_{k=1}^{n_i} a_k^i I_{A_k^i}$, where $A_k^i \in \sigma(T)$, i.e. there exists $B_k^i \in \mathcal{B}_2$ such that $A_k^i = T^{-1}(B_k^i)$. Define

$$Y = \limsup_{i\to\infty} \sum_{k=1}^{n_i} a_k^i I_{B_k^i}.$$

and notice for $\omega_1 \in \Omega_1$,

$$Y\big(T(\omega_1)\big) = \limsup_{i\to\infty} \sum_{k=1}^{n_i} a_k^i I_{B_k^i}\big(T(\omega_1)\big) = \limsup_{i\to\infty} \sum_{k=1}^{n_i} a_k^i I_{A_k^i}(\omega_1) = \limsup_{i\to\infty} X_i(\omega_1) = X(\omega_1).$$

Lastly, consider an arbitrary random variable X. Decompose $X = X^+ - X^-$, where $X^+ = \max\{X, 0\}$ and $X^- = -\min\{X, 0\}$. Noting that X^+ and X^- are both nonnegative random variables, by the above we have there exists random variables Y^+ and Y^- on $(\Omega_2, \mathcal{B}_2)$ such that for all $\omega_1 \in \Omega_1$, $X^+(\omega_1) = Y^+\big(T(\omega_1)\big)$ and $X^-(\omega_1) = Y^-\big(T(\omega_1)\big)$ Therefore,

$$Y\big(T(\omega_1)\big) = Y^+\big(T(\omega_1)\big) - Y^-\big(T(\omega_1)\big) = X^+(\omega_1) - X^-(\omega_1) = X(\omega_1)$$

for all $\omega_1 \in \Omega_1$. This completes the proof.

Problem 3.4.23

Suppose $X_1, ..., X_n$ are random variables on the probability space $(\Omega, \mathcal{B}, P)$ such that

$$P(\text{Ties}) = P\left(\bigcup_{1 \le i \ne j \le n} (X_i = X_j)\right) = 0.$$

Define the *relative rank* R_n of X_n among $X_1, ..., X_n$ to be

$$R_n = \begin{cases} \sum_{i=1}^n I(X_i \ge X_n), & \text{on } (\text{Ties})^c \\ 17, & \text{on Ties.} \end{cases}$$

Prove R_n is a random variable.

Solution: Notice that if a set A is measurable, then the indicator function I_A is a random variable since

$$\{I_A \le t\} = \begin{cases} \Omega, & t \ge 1 \\ A^c, & 0 \le t < 1 \\ \emptyset, & t < 0 \end{cases}$$

and $\Omega, A^c, \emptyset$ are all measurable. Also see that

$$\{X_i < X_j\} = \{X_i - X_j < 0\} \in \mathcal{B}$$

since the X_i's are random variables and the difference of random variables is a random variable. Therefore, since $\{X_i < X_j\} \in \mathcal{B}$, we have $I_{\{X_i < X_j\}}$ is a random variable. Again using the fact that the sum of random variables is a random variable, we have $\sum_{i=1}^n I(X_i \ge X_n)$ is a random variable. Also, since Ties is measurable, so is $(\text{Ties})^c$, and thus I_{Ties} and $I_{(\text{Ties})^c}$ are random variables. Therefore,

$$R_n = \sum_{i=1}^n I(X_i \ge X_n) I_{(\text{Ties})^c} + 17 \cdot I_{\text{Ties}}$$

is a random variable since the space of random variables is closed under addition, multiplication, and scalar multiplication.